Journal of Pure and Applied Algebra 80 (1992) 159–176
North-Holland

Commutative group rings of finite representation type

Lee Klingler*

Department of Mathematics, Florida Atlantic University, Boca Raton, FL 33431, USA

Communicated by C.A. Weibel
Received 21 May 1991
Revised 10 September 1991

Abstract

Klingler, L., Commutative group rings of finite representation type, Journal of Pure and Applied Algebra 80 (1992) 159–176.

Let $\mathbb{Z}_p$ denote the localization (but not the completion) of the integers at the prime p. Then, for finite groups G, the group ring $\mathbb{Z}_pG$ has finite representation type if and only if the p-Sylow subgroups of G are cyclic of order at most p^2. In this paper, we determine the possible ranks of indecomposable $\mathbb{Z}_pG$-lattices for finite abelian groups G for which $\mathbb{Z}_pG$ has finite representation type. In particular, for such groups G, we show that every indecomposable $\mathbb{Z}_pG$-lattice can be embedded as a sublattice of $\mathbb{Z}_pG^{(4)}$, but not, in general, as a sublattice of $\mathbb{Z}_pG^{(3)}$.

1. Introduction

Let R be a Dedekind domain with quotient field K, and let Λ be an R-order in the separable K-algebra $A = A_1 \oplus \cdots \oplus A_n$, where each A_i is simple. We say that Λ has *finite representation type* if there are only finitely many isomorphism classes of indecomposable Λ-lattices.

By a theorem of Jones [6], the R-order Λ has finite representation type if and only if, for each maximal ideal $\mathfrak{P}$ of R, the $\mathfrak{P}$-adic completion $\hat{\Lambda}_{\mathfrak{P}}$ has finite representation type. The proof, as given in [2, Theorem 33.2], relies on the fact that any subsemigroup of the additive semigroup C of t-tuples of nonnegative integers has only finitely many minimal elements. Here t is the total number of isomorphism classes of indecomposable lattices at the (finitely many) completions of Λ which are not maximal orders. (Of course, each coordinate of C corresponds to an isomorphism class of indecomposable lattices over one of these completions of Λ.) Although this proof shows that there are only finitely many isomorphism

Correspondence to: L. Klingler, Department of Mathematics, Florida Atlantic University, Boca Raton, FL 33431, USA.

* Research partially supported by a Humboldt Fellowship at the University of Essen.

classes of indecomposable lattices in this case, and hence there is a bound on their 'size', it gives no clue as to how many isomorphism classes of indecomposable Λ-lattices there are, or how large these indecomposables might be. In general, this is a very difficult question.

As a measure of the size of Λ-lattices, we define the *rank* of the Λ-lattice M (denoted by rank(M)) to be the n-tuple $(\alpha_1, \ldots, \alpha_n)$ of nonnegative integers such that α_i is the length of the A_i-module $A_i \cdot (K \otimes_R M)$. (Note that, if Λ is commutative, then the integers $\alpha_1, \ldots, \alpha_n$ are just the ranks of M at the minimal primes of Λ.) As is shown by examples in [5] and [7], given a class of R-orders of finite representation type, there might be no uniform bound on the ranks of indecomposable lattices for R-orders in the class. On the other hand, [10] shows that, if G is a cyclic group of square-free order, then every indecomposable $\mathbb{Z}G$-lattice embeds as a sublattice of $\mathbb{Z}G$, so that there is a uniform bound on the 'size' of $\mathbb{Z}G$-lattices for such groups G. As a second example, [1] shows that, if Λ is a commutative R-order of finite representation type, and if M is an indecomposable Λ-lattice of constant rank $(\alpha, \alpha, \ldots, \alpha)$, then $\alpha \leq 39$.

In this paper we show that, if G is a (finite) abelian group and p is a rational prime such that $\mathbb{Z}_pG$ is of finite representation type (where $\mathbb{Z}_p$ denotes the localization, not the completion, of $\mathbb{Z}$ at the prime p) then every indecomposable $\mathbb{Z}_pG$-lattice embeds as a sublattice of $\mathbb{Z}_pG^{(4)}$. Depending on the prime p and the exponent of G, there might exist an indecomposable $\mathbb{Z}_pG$-lattice which cannot be embedded as a sublattice of $\mathbb{Z}_pG^{(3)}$ (see Theorem 3.5).

In Section 2 we describe commutative group rings $\mathbb{Z}_pG$ of finite representation type as direct sums of pullbacks of certain semilocal principal ideal domains. We also describe the completions of these pullbacks and quote the appropriate results from [3] to obtain a description of indecomposable lattices over the completion $\hat{\mathbb{Z}}_pG$. In Section 3 we use the results of Section 2 for the completion $\hat{\mathbb{Z}}_pG$ together with an intricate combinatorial argument to determine the possible ranks of indecomposable $\mathbb{Z}_pG$-lattices. In Section 4 we give two examples to show that both the assumption that G is abelian and the use of the local coefficient ring $\mathbb{Z}_p$ are necessary in order to obtain this uniform bound. Our first example shows that, for any integer n, we can find a prime p and a (nonabelian) group G of square-free order such that $\mathbb{Z}_pG$ has an indecomposable lattice that cannot be embedded as a sublattice of $\mathbb{Z}_pG^{(n)}$. Our second example shows that, for any integer n, we can find a cyclic group G of cube-free order such that $\mathbb{Z}G$ has an indecomposable lattice that cannot be embedded as a sublattice of $\mathbb{Z}G^{(n)}$.

2. Commutative group rings

If n is a positive integer, we let ζ_n denote a primitive complex nth root of unity. We begin with a description of commutative, rational group algebras.

Proposition 2.1. *If G is a finite abelian group, then $\mathbb{Q}G \cong \bigoplus \mathbb{Q}(\zeta_{|H|})$, where H ranges over the set of all cyclic subgroups of G.*

Proof. This is a special case of [11, Theorem 1]. □

As an immediate corollary, we get a description of the integral group ring in certain local cases.

Corollary 2.2. *If p is prime and G is a finite abelian group with order relatively prime to p, then $\mathbb{Z}_pG \cong \bigoplus \mathbb{Z}_p(\zeta_{|H|})$, where H ranges over the set of all cyclic subgroups of G.*

Proof. By [12, Theorem 41.1], $\mathbb{Z}_pG$ is a maximal order in $\mathbb{Q}G$, and clearly $\bigoplus \mathbb{Z}_p(\zeta_{|H|})$ is the unique maximal $\mathbb{Z}_p$-order in $\bigoplus \mathbb{Q}(\zeta_{|H|})$. The corollary is then immediate from the proposition. □

Beginning with an abelian group G and a prime p, we can write G as the direct product of a subgroup Q with the p-Sylow subgroup P of G. Then, using the corollary, $\mathbb{Z}_pG \cong \mathbb{Z}_pQ \otimes_{\mathbb{Z}_p} \mathbb{Z}_pP \cong \bigoplus \mathbb{Z}_p(\zeta_{|H|})P$, where H ranges over the cyclic subgroups of G of order relatively prime to p. Clearly we can work with one summand at a time. So, let R be the ring $\mathbb{Z}_p(\zeta_n)$ for some integer n relatively prime to p. (In fact, we could take R to be any integral extension of $\mathbb{Z}_p$ in which p is unramified, but we do not need this generality here.) We consider the integral group ring RP for some abelian p-group P.

By [2, Theorem 33.6], RP has finite representation type if and only if P is cyclic of order at most p^2. First we consider the case where P is cyclic of order p, the trivial case having been handled by Corollary 2.2.

Proposition 2.3. *With notation as above, where $P = \langle x \rangle$ is cyclic of prime order p, the group ring RP is isomorphic to the pullback $\Omega = \{(\alpha_0, \alpha_1) \in R \oplus R(\zeta_p) \mid f_1(\alpha_0) = g_1(\alpha_1)\}$, where $f_1 : R \to R/pR$ is the canonical map, and $g_1 : R(\zeta_p) \to R/pR$ is the map induced by $g_1(\zeta_p) = 1 + pR$.*

Proof. The isomorphism $\theta : RP \to \Omega$ is induced by $\theta(x) = (1, \zeta_p)$. That θ is in fact an isomorphism onto is well known, and depends only upon the fact that p is unramified in R, so that $R \cap \mathbb{Z}_p(\zeta_p) = \mathbb{Z}_p$. See, for example, [8, Example 1.1]. □

Since p is unramified in R, the factor ring R/pR is a direct sum of fields, and hence RP is a *Dedekind-like ring* in the sense of [9]. Also from [9] we get the following corollary.

Corollary 2.4. *With notation as above, where P is cyclic of prime order p, the* *isomorphism classes of indecomposable RP-lattices consist of R, $R(\zeta_p)$, and pullbacks of R and $R(\zeta_p)$ mapping onto nontrivial summands of R/pR. In particular, every indecomposable RP-lattice can be embedded as a sublattice of RP.* $\square$

The interesting case, then, is the case where the p-Sylow subgroup P is cyclic of order p^2. For the remainder of this section we assume that P is cyclic of order p^2.

Proposition 2.5. *With notation as above, with $P = \langle x \rangle$ of order p^2 and $\bar{P} = P/\langle x^p \rangle$, the ring RP is isomorphic to the pullback $\Lambda = \{(\alpha, \alpha_2) \in R\bar{P} \oplus R(\zeta_{p^2}) \mid f_2(\alpha) = g_2(\alpha_2)\}$, where $f_2 : R\bar{P} \rightarrow R\bar{P}/pR\bar{P}$ is the canonical map, and $g_2 : R(\zeta_{p^2}) \rightarrow R\bar{P}/pR\bar{P}$ is the map induced by $g_2(\zeta_{p^2}) = \bar{x} + pR\bar{P}$.*

Proof. The isomorphism $\theta : RP \rightarrow \Lambda$ is induced by $\theta(x) = (\bar{x}, \zeta_{p^2})$. That θ is an isomorphism onto follows easily, as above, from the argument in [8, Example 1.1]. $\square$

Let us introduce some of the notation of [3]. Let Γ_0, Γ_1, and Γ_2 be hereditary R-orders, where R is a semi-local principal ideal domain with radical $\mathfrak{J}$, and suppose that there are maps $f_1 : \Gamma_0 \rightarrow \Gamma_0/\text{rad}\,\Gamma_0 = \Gamma_0/\mathfrak{J}\Gamma_0$ and $g_1 : \Gamma_1 \rightarrow \Gamma_1/\text{rad}\,\Gamma_1 \cong \Gamma_0/\text{rad}\,\Gamma_0$. Let Ω be the pullback of the maps f_1 and g_1; that is,

$$\Omega = \{(\alpha_0, \alpha_1) \in \Gamma_0 \oplus \Gamma_1 \mid f_1(\alpha_0) = g_1(\alpha_1)\}\,.$$

Suppose also that, for some integer n, there are maps $f_2 : \Omega \rightarrow \Omega/\mathfrak{J}^n\Omega$ and $g_2 : \Gamma_2 \rightarrow \Gamma_2/(\text{rad}\,\Gamma_2)^n \cong \Omega/\mathfrak{J}^n\Omega$. Let Λ be the pullback of f_2 and g_2, so that $\Lambda = \{(\alpha, \alpha_2) \in \Omega \oplus \Gamma_2 \mid f_2(\alpha) = g_2(\alpha_2)\}$. We call Λ a *special quasi-triad*. Combining Propositions 2.3 and 2.5, we get the following corollary.

Corollary 2.6. *With notation as above, where $P = \langle x \rangle$ is cyclic of order p^2, the order RP is isomorphic to the special quasi-triad constructed from R, $R(\zeta_p)$, and $R(\zeta_{p^2})$, where $f_1 : R \rightarrow R/pR$ is the canonical map, $g_1 : R(\zeta_p) \rightarrow R/pR$ is the map induced by $g_1(\zeta_p) = 1 + pR$, $f_2 : R\bar{P} \rightarrow R\bar{P}/pR\bar{P}$ is the canonical map (identifying $R\bar{P}$ with the pullback Ω of the maps f_1 and g_1), and $g_2 : R(\zeta_{p^2}) \rightarrow R\bar{P}/pR\bar{P} \cong R(\zeta_{p^2})/(\text{rad}\,R(\zeta_{p^2}))^p$ is the canonical map.*

Proof. We need only check that $R\bar{P}/pR\bar{P} \cong R(\zeta_{p^2})/(\text{rad}\,R(\zeta_{p^2}))^p$. This is done in [4, Proposition 4.4]. $\square$

In our notation, the ring $R = \mathbb{Z}_p(\zeta_n)$ need only be semi-local. Because the results of [3] apply to special quasi-triads over complete discrete valuation rings, we conclude this section by examining the completions of RP at the maximal ideals of R. Since the prime p is assumed to be unramified in R, we have that the radical of R is given by $\mathfrak{J} = pR = \mathfrak{P}_1 \cdot \ldots \cdot \mathfrak{P}_g$ for some distinct maximal ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_g$ of R. Moreover, the prime p is totally ramified in $\mathbb{Z}(\zeta_p)$ and $\mathbb{Z}(\zeta_{p^2})$, so that each $\mathfrak{P}_i$ is totally ramified in $R(\zeta_p)$ and $R(\zeta_{p^2})$. Let $\mathfrak{P}'_i$ be the maximal ideal of $R(\zeta_p)$ containing $\mathfrak{P}_i$, and let $\mathfrak{P}''_i$ be the maximal ideal of $R(\zeta_{p^2})$ containing $\mathfrak{P}'_i$.

Now, the p-adic completion of RP is given by $\hat{R}_p P \cong \bigoplus_{i=1}^g \hat{R}_{\mathfrak{P}_i} P$. Also, $R/pR \cong \bigoplus_{i=1}^g R/\mathfrak{P}_i$, so that $(\hat{R}/p\hat{R})_{\mathfrak{P}_i} \cong R/\mathfrak{P}_i$. Similarly,

$$R(\zeta_{p^2})/(1-\zeta_{p^2})^p \cdot R(\zeta_{p^2}) \cong \bigoplus_{i=1}^{g} R(\zeta_{p^2})/(\mathfrak{P}''_i)^p ,$$

so that

$$(\hat{R}(\zeta_{p^2})/(1-\zeta_{p^2})^p \cdot \hat{R}(\zeta_{p^2}))_{\mathfrak{P}_i} \cong R(\zeta_{p^2})/(\mathfrak{P}''_i)^p .$$

From [4, Proposition 4.2] we get the following description of the p-adic completion of RP.

Proposition 2.7. *With notation as above, where P is cyclic of order p^2, the p-adic completion $\hat{R}_p P \cong \bigoplus_{i=1}^g \hat{R}_{\mathfrak{P}_i} P$, where each $\hat{R}_{\mathfrak{P}_i} P$ is the special quasi-triad constructed from $\hat{R}_{\mathfrak{P}_i}$, $\hat{R}(\zeta_p)_{\mathfrak{P}'_i}$, and $\hat{R}(\zeta_{p^2})_{\mathfrak{P}''_i}$ formed by the maps*

$$(f_1)_i : \hat{R}_{\mathfrak{P}_i} \to R/\mathfrak{P}_i , \qquad (g_1)_i : \hat{R}(\zeta_p)_{\mathfrak{P}'_i} \to R/\mathfrak{P}_i ,$$

$$(f_2)_i : \hat{R}_{\mathfrak{P}_i}\bar{P} \to R\bar{P}/\mathfrak{P}_i\bar{P} ,$$

$$(g_2)_i : \hat{R}(\zeta_{p^2})_{\mathfrak{P}''_1} \to R\bar{P}/\mathfrak{P}_i\bar{P} \cong R(\zeta_{p^2})/(\mathfrak{P}''_i)^p . \qquad \square$$

3. Indecomposable $\mathbb{Z}_pG$-lattices

Throughout this section we let Λ denote the special quasi-triad described in Corollary 2.6, where $R = \mathbb{Z}_p(\zeta_n)$ for some integer n relatively prime to p. (That is, Λ is an indecomposable summand of the group ring $\mathbb{Z}_pG$, where G is an abelian group with cyclic p-Sylow subgroup of order p^2, and with a cyclic subgroup of order n.) Writing $pR = \mathfrak{P}_1 \cdot \ldots \cdot \mathfrak{P}_g$, from Proposition 2.7, we get that the p-adic completion $\hat{\Lambda}_p \cong \bigoplus_{i=1}^g \hat{\Lambda}_{\mathfrak{P}_i}$, where each $\hat{\Lambda}_{\mathfrak{P}_i}$ is a special quasi-triad constructed from the $\mathfrak{P}_i$-adic completions $\hat{R}_{\mathfrak{P}_i}$, $\hat{R}(\zeta_p)_{\mathfrak{P}'_i}$, and $\hat{R}(\zeta_{p^2})_{\mathfrak{P}''_i}$.

As in the Introduction, given a $\hat{\Lambda}_{\mathfrak{P}_i}$-lattice M, we write $\operatorname{rank}(M) = (\alpha_0, \alpha_1, \alpha_2)$, where α_0 is the rank of $\hat{R}_{\mathfrak{P}_i} \cdot M$ as a free $\hat{R}_{\mathfrak{P}_i}$-lattice, α_i is the rank of $\hat{R}(\zeta_p)_{\mathfrak{P}'_i} \cdot M$ as a free $\hat{R}(\zeta_p)_{\mathfrak{P}'_i}$-lattice, and α_2 is the rank of $\hat{R}(\zeta_{p^2})_{\mathfrak{P}''_i} \cdot M$ as a free $\hat{R}(\zeta_{p^2})_{\mathfrak{P}''_i}$-lattice. If M is a Λ-lattice, clearly $\operatorname{rank}(\hat{M}_{\mathfrak{P}_i})$ is the same for all of the $\mathfrak{P}_i$-adic completions $\hat{M}_{\mathfrak{P}_i}$, $1 \le i \le g$; we define $\operatorname{rank}(M)$ to be this rank. From [3, Theorem 1.9], we record the possible ranks of indecomposable $\hat{\Lambda}_{\mathfrak{P}_i}$-lattices.

Proposition 3.1. *If $p = 2$, then, for each index i, the possible ranks of indecomposable $\hat{\Lambda}_{\mathfrak{P}_i}$-lattices are* $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ *and* $(1,1,1)$. *If $p > 2$, then, in addition to the above list of possible ranks, $\hat{\Lambda}_{\mathfrak{P}_i}$ has indecomposable lattices of rank* $(2,1,1)$. □

One of our main tools is the following standard fact. (See, for example, [12, Theorem 4.26].)

Proposition 3.2. *Let S be some subset of $\{1, 2, \ldots, g\}$ and suppose that M_i is a $\hat{\Lambda}_{\mathfrak{P}_i}$-lattice for each index $i \in S$. Then there exists a Λ-lattice M such that $\hat{M}_{\mathfrak{P}_i} \cong M_i$ for each index $i \in S$ if and only if* $\operatorname{rank}(M_i)$ *is the same for all indices $i \in S$.* □

For ranks $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $\beta = (\beta_0, \beta_1, \beta_2)$, we write $\alpha \leq \beta$ if $\alpha_k \leq \beta_k$ for $k = 0,1,2$, and we write $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

Corollary 3.3. *For each nonzero triple of integers $\beta < \operatorname{rank}(M)$, the Λ-lattice M has a summand of rank β if and only if, for all indices i, $1 \leq i \leq g$, the $\mathfrak{P}_i$-adic completion $\hat{M}_{\mathfrak{P}_i}$ has a summand of rank β.*

Proof. Since R is semi-local, given Λ-lattices M, N_1, and N_2, we have $M \cong N_1 \oplus N_2$ if and only if $\hat{M}_{\mathfrak{P}_i} \cong (\hat{N}_1)_{\mathfrak{P}_i} \oplus (\hat{N}_2)_{\mathfrak{P}_i}$ for each maximal ideal $\mathfrak{P}_i$ of R. The result now follows immediately from Proposition 3.2. □

Combining Proposition 3.1 and Corollary 3.3, we get the following corollary.

Corollary 3.4. *The order Λ has an indecomposable lattice of rank α if and only if, for each index i, $1 \leq i \leq g$, we can write α as a sum $\beta_{i,1} + \cdots + \beta_{i,m_i}$ of some sequence of ranks from Proposition* 3.1, *in such a way that, for each nonzero $\beta < \alpha$, there exists an index i for which β is not a sum of some sub-sequence of $\beta_{i,1}, \ldots, \beta_{i,m_i}$.* □

Our main result is to determine the possible ranks of indecomposable Λ-lattices. The surprising fact is that there is a bound on the rank of indecomposable Λ-lattices, independent of the number g of maximal ideals of R.

Theorem 3.5. *With notation as above, where p is prime and $pR = \mathfrak{P}_1 \cdot \ldots \cdot \mathfrak{P}_g$, let X be an indecomposable Λ-lattice.*

(i) *If $p = 2$ and $g = 1$, then* $\operatorname{rank}(X)$ *is one of* $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, *or* $(1,1,1)$. (*In particular, X embeds in Λ.*) *In addition, Λ has indecomposable lattices of each of these ranks.*

(ii) *If $p > 2$ and $g = 1$, then either* $\operatorname{rank}(X)$ *is one of the ranks in* (i), *or* $\operatorname{rank}(X) = (2,1,1)$. (*In particular, X embeds in $\Lambda \oplus \Lambda$.*) *As above, Λ has indecomposable lattices of each of these ranks.*

(iii) *If* $p=2$ *and* $g \geq 2$, *then either* $\operatorname{rank}(X)$ *is one of the above ranks, or* $\operatorname{rank}(X)$ *is one of* $(1,2,1)$, $(1,1,2)$, *or* $(2,2,2)$. (*In particular, X embeds in* $\Lambda \oplus \Lambda$.) *As above, Λ has indecomposable lattices of each of these ranks.*

(iv) *If* $p>2$ *and* $g=2$, *then either* $\operatorname{rank}(X)$ *is one of the above ranks, or* $\operatorname{rank}(X)$ *is one of* $(2,2,1)$, $(2,1,2)$, $(2,3,1)$, *or* $(2,1,3)$. (*In particular, X embeds in* $\Lambda \oplus \Lambda \oplus \Lambda$.) *As above, Λ has indecomposable lattices of each of these ranks.*

(v) *If* $p>2$ *and* $g \geq 3$, *then either* $\operatorname{rank}(X)$ *is one of the above ranks, or* $\operatorname{rank}(X)$ *is one of* $(2,3,2)$, $(2,2,3)$, $(3,3,2)$, $(3,2,3)$, $(2,4,2)$, $(2,2,4)$, $(4,4,2)$, *or* $(4,2,4)$. (*In particular, X embeds in* $\Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda$.) *As above, Λ has indecomposable lattices of each of these ranks.*

Proof. If $g=1$, then the p-adic completion of the quotient field of R is again a field. In this case, it follows from [2, Theorem 30.18] that indecomposable Λ-lattices remain indecomposable at the p-adic completion. Parts (i) and (ii) then follow from Proposition 3.1.

Next we show that there exist appropriate indecomposable lattices of each of the ranks indicated for cases (iii) through (v).

Suppose, first, that $p=2$ and $g \geq 2$, and write $2R = \mathfrak{P}_1 \cdot \mathfrak{P}_2 \cdot \ldots \cdot \mathfrak{P}_g$. Clearly Λ has indecomposable lattices of each of the ranks listed in (i). By a slight abuse of notation, we can define the $\hat{\Lambda}_{\mathfrak{P}_1}$-lattice $X_1 = (1,1,1) \oplus (1,0,0)$. (That is, X_1 is the direct sum of an indecomposable $\hat{\Lambda}_{\mathfrak{P}_1}$-lattice of rank $(1,1,1)$ and an indecomposable $\hat{\Lambda}_{\mathfrak{P}_1}$-lattice of rank $(1,0,0)$.) Similarly, we can define the $\hat{\Lambda}_{\mathfrak{P}_2}$-lattice $X_2 = (1,0,1) \oplus (1,1,0)$. By Proposition 3.2 there exists a Λ-lattice X such that $\hat{X}_{\mathfrak{P}_1} \cong X_1$ and $\hat{X}_{\mathfrak{P}_2} \cong X_2$, and it follows from Corollary 3.3 that X is an indecomposable Λ-lattice of rank $(2,1,1)$. Analogous constructions yield indecomposable Λ-lattices of ranks $(1,2,1)$ and $(1,1,2)$. Similarly, by Proposition 3.2, there exists an indecomposable Λ-lattice Y with completions $\hat{Y}_{\mathfrak{P}_1} \cong (1,1,1) \oplus (1,1,1)$ and $\hat{Y}_{\mathfrak{P}_2} \cong (1,1,0) \oplus (1,0,1) \oplus (0,1,1)$, so that, by Corollary 3.3, Y is an indecomposable Λ-lattice of rank $(2,2,2)$. This completes the proof that, if $p=2$ and $g \geq 2$, then Λ has indecomposable lattices of each of the ranks listed in (i) through (iii).

Suppose, next, that $p>2$ and $g=2$, and write $pR = \mathfrak{P}_1 \cdot \mathfrak{P}_2$. Then each of the above constructions work for Λ in this case. In addition, we can find an indecomposable Λ-lattice X of rank $(2,2,1)$ such that $\hat{X}_{\mathfrak{P}_1} \cong (2,1,1) \oplus (0,1,0)$ and $\hat{X}_{\mathfrak{P}_2} \cong (1,1,1) \oplus (1,1,0)$. (Of course, a similar construction will yield an indecomposable Λ-lattice of rank $(2,1,2)$.) Similarly, we can find an indecomposable Λ-lattice Y of rank $(2,3,1)$ such that $\hat{X}_{\mathfrak{P}_1} \cong (2,1,1) \oplus (0,1,0) \oplus (0,1,0)$ and $\hat{X}_{\mathfrak{P}_2} \cong (1,1,0) \oplus (1,1,0) \oplus (0,1,1)$. (Again, a similar construction will yield an indecomposable Λ-lattice of rank $(2,1,3)$.) This completes the proof that, if $p>2$ and $g=2$, then Λ has indecomposable lattices of each of the ranks listed in (i) through (iv).

Finally, suppose that $p>2$ and $g \geq 3$, and write $pR = \mathfrak{P}_1 \cdot \mathfrak{P}_2 \cdot \mathfrak{P}_3 \cdot \ldots \cdot \mathfrak{P}_g$.

Again, each of the above constructions work for Λ in this case. As above, we can find an indecomposable Λ-lattice X of rank $(2,3,2)$ such that

$$\hat{X}_{\mathfrak{P}_1} \cong (2,1,1) \oplus (0,1,1) \oplus (0,1,0)\,,$$

$$\hat{X}_{\mathfrak{P}_2} \cong (1,1,1) \oplus (1,1,1) \oplus (0,1,0)\,,$$

and

$$\hat{X}_{\mathfrak{P}_3} \cong (1,1,1) \oplus (1,1,0) \oplus (0,1,1)$$

(and similarly for the rank $(2,2,3)$). We can find an indecomposable Λ-lattice Y of rank $(3,3,2)$ such that

$$\hat{Y}_{\mathfrak{P}_1} \cong (2,1,1) \oplus (0,1,1) \oplus (1,1,0)\,,$$

$$\hat{Y}_{\mathfrak{P}_2} \cong (1,1,1) \oplus (1,1,1) \oplus (1,1,0)\,,$$

and

$$\hat{Y}_{\mathfrak{P}_3} \cong (2,1,1) \oplus (1,1,1) \oplus (0,1,0)$$

(and similarly for the rank $(3,2,3)$). We can find an indecomposable Λ-lattice Z of rank $(4,4,2)$ such that

$$\hat{Z}_{\mathfrak{P}_1} \cong (2,1,1) \oplus (2,1,1) \oplus (0,1,0) \oplus (0,1,0)\,,$$

$$\hat{Z}_{\mathfrak{P}_2} \cong (1,1,1) \oplus (1,1,1) \oplus (1,1,0) \oplus (1,1,0)\,,$$

and

$$\hat{Z}_{\mathfrak{P}_3} \cong (2,1,1) \oplus (0,1,1) \oplus (1,1,0) \oplus (1,1,0)$$

(and similarly for the rank $(4,2,4)$). Finally, we can find an indecomposable Λ-lattice W of rank $(2,4,2)$ such that

$$\hat{W}_{\mathfrak{P}_1} \cong (2,1,1) \oplus (0,1,1) \oplus (0,1,0) \oplus (0,1,0)\,,$$

$$\hat{W}_{\mathfrak{P}_2} \cong (1,1,1) \oplus (1,1,1) \oplus (0,1,0) \oplus (0,1,0)\,,$$

and

$$\hat{W}_{\mathfrak{P}_3} \cong (1,1,0) \oplus (1,1,0) \oplus (0,1,1) \oplus (0,1,1)\,,$$

(and similarly for the rank $(2,2,4)$). This completes the proof that, if $p>2$ and

$g \geq 3$, then Λ has indecomposable lattices of each of the ranks listed in (i) through (v).

To complete the proof, we show that, if $p > 2$ and $g \geq 3$, then the ranks listed in (i) through (v) are the only possible ranks of indecomposable Λ-lattices. (We leave as a tedious but straightforward exercise the verification that, if $p = 2$, then Λ has no indecomposable lattices with ranks as in (iv) or (v), while if $g = 2$, then Λ has no indecomposable lattices with ranks as in (v).) The proof of (v) is a long combinatorial argument, which we break down into a number of steps.

Step 1. We claim that, if the Λ-lattice X has rank (r, r, r) for some integer $r \geq 2$, then X contains a summand of rank $(2, 2, 2)$.

By Corollary 3.3, it suffices to prove the claim for the $\hat{\Lambda}_{\mathfrak{P}_i}$-lattice $X_i = \hat{X}_{\mathfrak{P}_i}$ of rank (r, r, r). The proof is by induction on r. The case $r = 2$ is of course trivial, so suppose that $r > 2$. If X_i contains a (not necessarily indecomposable) summand of rank $(1, 1, 1)$, then it contains a complementary summand of rank $(r-1, r-1, r-1)$, so that, by induction, it contains a summand of rank $(2, 2, 2)$.

Thus, suppose that X_i contains no summand of rank $(1, 1, 1)$. If X_i contains a (not necessarily indecomposable) summand of rank $(2, 1, 1)$, then a complementary summand of rank $(r-2, r-1, r-1)$ must also contain an indecomposable summand of rank (a, b, c) in which $b > a$. Checking the list of ranks of indecomposable summands of $\hat{\Lambda}_{\mathfrak{P}_i}$-lattices given in Proposition 3.1, we see that this summand must contain an indecomposable summand of rank $(0, 1, 1)$ or $(0, 1, 0)$. If it contains an indecomposable summand of rank $(0, 1, 1)$, then it contains a summand of rank $(2, 2, 2)$, and we are done. Otherwise, it must also contain a summand of rank (a, b, c) in which $c > a$, so, since it contains no indecomposable summand of rank $(0, 1, 1)$, it must contain a summand of rank $(0, 0, 1)$, and we are done. Thus, if X_i contains a (not necessarily indecomposable) summand of rank $(1, 1, 1)$ or rank $(2, 1, 1)$, then it must contain a summand of rank $(2, 2, 2)$.

Suppose , then, that X_i contains no summands of rank $(1, 1, 1)$ or $(2, 1, 1)$; we show that this assumption leads to a contradiction. Suppose that X_i contains a summand of rank $(1, 1, 0)$. Then it cannot contain a summand of rank $(0, 0, 1)$ (since it contains no summand of rank $(1, 1, 1)$), and it cannot contain a summand of rank $(1, 0, 1)$ (since it contains no summand of rank $(2, 1, 1)$). So, the only indecomposable summands of X_i of nonzero rank in the third coordinate must be of rank $(0, 1, 1)$. But since X_i contains a summand of rank $(1, 1, 0)$, this implies that the rank in the second coordinate must be larger than that in the third coordinate, contradicting the fact that X_i has rank (r, r, r). Similarly, if X_i contains a summand of rank $(1, 0, 1)$, we get a contradiction. Thus, the only indecomposable summands of X_i of nonzero rank in the first coordinate must be of rank $(1, 0, 0)$. As argued above, X_i must contain either an indecomposable summand of rank $(0, 1, 1)$ or indecomposable summands of rank $(0, 1, 0)$ and $(0, 0, 1)$, either of which leads to a summand of rank $(1, 1, 1)$, contrary to assumption.

This completes the proof of Step 1. Note that we have shown that, if X is indecomposable of rank (r, r, r), then X has rank $(1,1,1)$ or $(2,2,2)$, both of which are on the list of acceptable ranks in (i) through (v) above.

Step 2. We claim that, if X is a Λ-lattice of rank (a, b, c), and if one or more of a, b, or c is 0, then X contains a summand whose rank is one of $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, or $(0,1,1)$.

The proof is straightforward. For example, if $a=0$ but $b>0$ and $c>0$, then as above, for each index i, $\hat{X}_{\mathfrak{P}_i}$ must contain either an indecomposable summand of rank $(0,1,1)$ or indecomposable summands of rank $(0,1,0)$ and $(0,0,1)$. Either way, by the Corollary 3.3, X contains a summand of rank $(0,1,1)$. Thus, if X is indecomposable of rank (a, b, c), and if one or more of a, b, or c is 0, then the rank of X must be one of $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, or $(0,1,1)$.

Step 3. We claim that, if X is a Λ-lattice of rank (a, b, c) such that $a<b$ and $a<c$, then X contains a summand of rank $(0,1,1)$.

As above, since $a<b$ and $a<c$, for each index i, $\hat{X}_{\mathfrak{P}_i}$ must contain either an indecomposable summand of rank $(0,1,1)$ or indecomposable summands of rank $(0,1,0)$ and $(0,0,1)$. Either way, by Corollary 3.3, X contains a summand of rank $(0,1,1)$. Thus, if X is indecomposable of rank (a, b, c) with $a<b$ and $a<c$, then X has rank $(0,1,1)$.

Step 4. We claim that, if X is a Λ-lattice of rank (a, b, c) such that $a>b>0$ and $a>c>0$, then X contains a summand of rank $(2,1,1)$.

As above, it suffices to prove the claim for each completion $X_i=\hat{X}_{\mathfrak{P}_i}$. Suppose, by way of contradiction, that, for some index i, X_i did not contain a (decomposable or indecomposable) summand of rank $(2,1,1)$. If X_i also contained no summand of rank $(1,0,0)$, then, since $a>b$ and $a>c$, X_i would have to contain summands of rank $(1,1,0)$ and $(1,0,1)$, contradicting the fact that X_i contains no summand of rank $(2,1,1)$. Thus, X_i must contain a summand of rank $(1,0,0)$, and hence a complementary summand of rank $(a-1, b, c)$. If $a-1>b$ and $a-1>c$, then by induction on a we get that X_i contains a summand of rank $(2,1,1)$, contradicting our assumption. Thus, we can assume that $a-1=b \geq c$. (The case where $a-1=c \geq b$ is of course symmetric.) Arguing as above, if $b>c$, then the complementary summand of rank (b, b, c) must contain either an indecomposable summand of rank $(1,1,0)$ or indecomposable summands of rank $(1,0,0)$ and $(0,1,0)$, since it contains no summand of rank $(2,1,1)$. Thus this complementary summand itself contains a summand of rank $(b-1, b-1, c)$ so that, by induction on b, it contains a summand of rank (c, c, c). We must have $c>1$, since X_i contains no summand of rank $(2,1,1)$, and hence by Step 1 above, X_i must contain a summand of rank $(2,2,2)$. But this summand of rank $(2,2,2)$ cannot contain a summand of rank $(2,1,1)$, so that, by a simple argument, it must contain a summand of rank $(1, 1, 1)$. Therefore, X_i must contain a summand of rank $(2, 1, 1)$, the final contradiction. This completes the proof of Step 4.

Note that Steps 2 and 4 together show that, if X is indecomposable of rank (a, b, c) with $a > b$ and $a > c$, then X has rank $(2, 1, 1)$ or $(1, 0, 0)$.

Step 5. Combining the results of Steps 1 through 4, if X is an indecomposable Λ-lattice of rank (a, b, c), then the theorem is proven in case $a = b = c$, or if any of a, b, or c is 0, or if $a < b$ and $a < c$, or if $a > b$ and $a > c$. Thus we assume, for the remainder of the proof, that X is a Λ-lattice of rank (a, b, c) in which $0 < c \leq a \leq b$ and such that $a \neq b$ or $a \neq c$. (Given the list of indecomposable lattices over the completion $\hat{\Lambda}_{\mathfrak{P}_i}$, the argument is clearly symmetric in the case where $0 < b \leq a \leq c$.) In this step, we assume that $0 < c < a = b$, and we claim that X has a summand of rank $(2, 2, 1)$, $(3, 3, 2)$, or $(4, 4, 2)$.

First suppose that $X_i = \hat{X}_{\mathfrak{P}_i}$ is a completion of X that does not contain a summand of rank $(2, 2, 1)$. As our first 'subclaim' we claim that X_i consists of the direct sum of $c/2$ summands of rank $(2, 2, 2)$ with $a - c$ summands of rank $(1, 1, 0)$. (Note that this implies that c is even.)

The proof of the first subclaim is by induction on $a + b + c$. If X_i contains a summand of rank $(2, 1, 1)$, then by assumption, a complementary summand contains no summand of rank $(0, 1, 0)$. Since $b = a$, it must contain a summand of rank $(0, 1, 1)$, and hence X_i must contain a summand of rank $(2, 2, 2)$. If $c > 2$, then the subclaim follows by induction, using the complementary summand of rank $(a - 2, a - 2, c - 2)$. If $c = 2$, then X_i contains a complementary summand of rank $(a - 2, a - 2, 0)$, with $a - 2 > 0$, and an easy induction on $a - 2$ shows that this summand is the direct sum of $a - 2$ lattices each of rank $(1, 1, 0)$.

Suppose, then, that X_i contains no summand of rank $(2, 1, 1)$, either. Since $a > c$, X_i must contain a summand of rank $(1, 1, 0)$ or a summand of rank $(1, 0, 0)$. Suppose that it contains a summand of rank $(1, 1, 0)$, so that it also contains a complementary summand of rank $(a - 1, a - 1, c)$. If $a - 1 > c$, then the subclaim follows by induction, using the summand of rank $(a - 1, a - 1, c)$. If $a - 1 = c$, then by Step 1, the summand of rank (c, c, c) is a direct sum of summands of rank $(2, 2, 2)$ with summands of rank $(1, 1, 1)$. Since X_i contains a summand of rank $(1, 1, 0)$ but none of rank $(2, 2, 1)$, it contains no summand of rank $(1, 1, 1)$, and again the subclaim follows by induction.

Finally, suppose that X_i has no summand of rank $(2, 2, 1)$, $(2, 1, 1)$, or $(1, 1, 0)$. Since $a > c$, X_i must contain a summand of rank $(1, 0, 0)$, but since $b > c$ also, X_i must contain a summand of rank $(0, 1, 0)$, contradiction. This completes the proof of the first subclaim. (Note that it follows that, if X_i contains no summand of rank $(2, 2, 1)$, then it must have a summand of rank $(3, 3, 2)$.)

Suppose that $X_i = \hat{X}_{\mathfrak{P}_i}$ is a completion of X that contains no summand of rank $(3, 3, 2)$, and suppose that c is even. As our second 'subclaim' we claim that $a > c + 1$ and that X_i contains a summand of rank $(4, 4, 2)$. By the previous subclaim, X_i must contain a summand of rank $(2,2,1)$, so that it contains a complementary summand of rank $(a-2, a-2, c-1)$. If $a-2=c-1$, then since $c-1$ is odd, by Step 1 this summand would contain a summand of rank $(1,1,1)$, so that X_i would contain a summand of rank $(3,3,2)$, contrary to hypothesis. Thus $a-2>c-1$ (and hence $a>c+1$). Again by the previous subclaim, since $c-1$ is odd, this summand of rank $(a-2, a-2, c-1)$ must contain a summand of rank $(2,2,1)$, from which the second subclaim follows.

We are now ready to prove the claim. Suppose that X contains no summand of rank $(2,2,1)$ or of rank $(3,3,2)$. Using Corollary 3.3, by the first subclaim, c is even, and by the second subclaim, $a>c+1$. Consider a completion $X_i = \hat{X}_{\mathfrak{P}_i}$. By the first subclaim, if X_i contains no summand of rank $(2,2,1)$, then it contains a summand of rank $(4,4,2)$. On the other hand, as in the proof of the second subclaim, if X_i contains a summand of rank $(2,2,1)$, then it contains a complementary summand of rank $(a-2, a-2, c-1)$, and, by the first subclaim, since $a-2>c-1$ and $c-1$ is odd, this summand contains a summand of rank $(2,2,1)$ also. Thus, by Corollary 3.3, X contains a summand of rank $(4,4,2)$.

This completes the proof of Step 5. Note that we have shown that, if X is indecomposable of rank (a,a,c) with $a>c>0$, then X has rank $(2,2,1)$, $(3,3,2)$, or $(4,4,2)$, all of which are on the list of acceptable ranks in (i) through (v) above.

Step 6. Suppose that X is a Λ-lattice of rank (a,b,c) in which $0<c=a<b$; we claim that X has a summand of rank $(1,2,1)$, $(2,3,2)$, or $(2,4,2)$.

First suppose that $X_i = \hat{X}_{\mathfrak{P}_i}$ is a completion of X that does not contain a summand of rank $(1,2,1)$. As a 'subclaim' we claim that X_i consists of the direct sum of $a/2$ summands of rank $(2,2,2)$ with $b-a$ summands of rank $(0,1,0)$. (Note that this implies that a is even.) The proof is similar to that of the first subclaim of Step 5. Since X_i contains no summand of rank $(1,2,1)$ it cannot contain both a summand of rank $(1,1,0)$ and a summand of rank $(0,1,1)$. Since $b>a=c$, clearly X_i must contain a summand of rank $(0,1,0)$. By induction on $b-a$, it follows that X_i consists of the direct sum of $b-a$ summands of rank $(0,1,0)$ with a summand of rank (a,a,a). From Step 1, it follows that this summand is the direct sum of summands of rank $(2,2,2)$ with summands of rank $(1,1,1)$. Since X_i contains a summand of rank $(0,1,0)$ but none of rank $(1,2,1)$, it cannot have a summand of rank $(1,1,1)$. This proves the subclaim.

Now suppose that X contains no summand of rank $(1,2,1)$. By the Corollary 3.3 and the above subclaim, a must be even. There are two cases to consider here. Suppose, first, that $b>a+1$. For each completion X_i that contains no summand of rank $(1,2,1)$, the subclaim shows that X_i contains a summand of rank $(2,4,2)$. On the other hand, each completion X_i that contains a summand of rank $(1,2,1)$ must also contain a complementary summand of rank $(a-1, b-2, a-1)$, with $b-2>a-1$, and $a-1$ odd. By the subclaim above, this summand must also contain a summand of rank $(1,2,1)$, so that X_i contains a summand of rank

$(2, 4, 2)$ in this case as well. Thus, by Corollary 3.3, X contains a summand of rank $(2, 4, 2)$.

Still assuming that X has no summand of rank $(1, 2, 1)$, suppose that $b = a + 1$, so that X has rank $(a, a+1, a)$, with a even. If X_i is a completion containing a summand of rank $(0, 1, 0)$, then it contains a complementary summand of rank (a, a, a), with $a \geq 2$. By Step 1, this summand contains a summand of rank $(2, 2, 2)$, so that X_i contains a summand of rank $(2, 3, 2)$. If X_i is a completion containing no summand of rank $(0, 1, 0)$, then, since $b > a = c$, X_i must contain summands of rank $(1, 1, 0)$ and $(0, 1, 1)$, and hence a summand of rank $(1, 2, 1)$. Again, X_i must contain a complementary summand of rank $(a-1, a-1, a-1)$ with $a - 1$ odd. By Step 1, this summand must contain a summand of rank $(1, 1, 1)$ (using induction and the fact that $a - 1$ is odd), so that X_i contains a summand of rank $(2, 3, 2)$ in this case also. Thus, by Corollary 3.3, X contains a summand of rank $(2, 3, 2)$.

This completes the proof of Step 6. Note that we have shown that, if X is indecomposable of rank (a, b, a) with $b > a > 0$, then X has rank $(1, 2, 1)$, $(2, 3, 2)$, or $(2, 4, 2)$, all of which are on the list of acceptable ranks in (i) through (v) above.

Step 7. We can now assume that X is a Λ-lattice of rank (a, b, c) in which $0 < c < a < b$. In this step, we assume in addition that $b > c + 2$, so that $b > a + 1$ or $a > c + 1$. We claim that X contains a summand of rank $(2, 3, 1)$.

By Corollary 3.3, it suffices to prove the claim for the $\hat{\Lambda}_{\mathfrak{P}_i}$-lattice $X_i = \hat{X}_{\mathfrak{P}_i}$ of rank (a, b, c). The proof is by induction on c. Suppose, first, that $c = 1$. If X_i contains a summand of rank $(2, 1, 1)$, then it contains a complementary summand of rank $(a-2, b-1, 0)$. But $b > a$ implies that $b - 1 \geq (a-2) + 2$, so that this complementary summand must contain a summand of rank $(0, 2, 0)$, and hence X_i contains a summand of rank $(2, 3, 1)$. Similarly, if X_i contains a summand of rank $(1, 1, 1)$, then it contains a complementary summand of rank $(a-1, b-1, 0)$. Since $b - 1 > a - 1 > 0$, this complementary summand must contain a summand of rank $(1, 2, 0)$, so that X_i contains a summand of rank $(2, 3, 1)$. If X_i contains a summand of rank $(0, 1, 1)$, then it contains a complementary summand of rank $(a, b-1, 0)$. Since $b - 1 \geq a \geq 2$, this complementary summand must contain a summand of rank $(2, 2, 0)$, and again X_i contains a summand of rank $(2, 3, 1)$. By a similar argument, if X_i contains a summand of rank $(1, 0, 1)$ or a summand of rank $(0, 0, 1)$, then it contains a summand of rank $(2, 3, 1)$. Since $c = 1$, given the list of possible ranks of indecomposable Λ_i-lattices, X_i must contain a summand of one of the ranks $(2, 1, 1)$, $(1, 1, 1)$, $(0, 1, 1)$, $(1, 0, 1)$, or $(0, 0, 1)$. Therefore, the proof of Step 7 is complete in the case $c = 1$.

Suppose now that $c > 1$. If X_i contains a summand of rank $(1, 1, 1)$, $(1, 0, 1)$, or $(0, 1, 1)$, then by induction on c, its complementary summand contains a summand of rank $(2, 3, 1)$.

Suppose that X_i contains no summand of rank $(1, 1, 1)$, $(1, 0, 1)$, or $(0, 0, 1)$ but does contain a summand of rank $(2, 1, 1)$, so that it contains a complementary summand of rank $(a-2, b-1, c-1)$. If $a-2>c-1$, then by induction on c, this summand contains a summand of rank $(2, 3, 1)$. Suppose, instead, that $a-2=c-1$. Then, the summand of X_i of rank $(2, 1, 1)$ has a complementary summand of rank $(c-1, b-1, c-1)$, where $b-1 \geq (c-1)+3$. If this complementary summand contains a summand of rank $(0, 2, 0)$, then X_i contains a summand of rank $(2, 3, 1)$. Otherwise, since $b-1 \geq (c-1)+3$ and $c>1$, the complementary summand must contain a summand of rank $(2, 2, 0)$ and a summand of rank $(0, 1, 1)$, and hence a summand of rank $(2, 3, 1)$.

Finally, we can suppose that X_i does not contain a summand of rank $(1, 1, 1)$, $(1, 0, 1)$, $(0, 0, 1)$, or $(2, 1, 1)$. Then it must contain a summand of rank $(0, 1, 1)$, and a complementary summand of rank $(a, b-1, c-1)$. Since X_i contains no summand of rank $(1, 1, 1)$, this complementary summand contains no summand of rank $(1, 0, 0)$. But $a>c-1$, so that it must contain a summand of rank $(1, 1, 0)$ and a complementary summand of rank $(a-1, b-2, c-1)$. Since $a-1>c-1>0$, this complementary summand must also contain a summand of rank $(1, 1, 0)$, so that X_i contains a summand of rank $(2, 3, 1)$.

This completes the proof of Step 7. Note that we have shown that, if X is indecomposable of rank (a, b, c) with $0<c<a<b$ and $b>c+2$, then X has rank $(2, 3, 1)$, which is on the list of acceptable ranks in (i) through (v) above.

Step 8. Finally, we can suppose that X is a Λ-lattice of rank $(c+1, c+2, c)$, where $c>0$.

If $c=1$, then there is of course nothing to prove.

If $c=2$, we claim that X contains a summand of rank $(1, 2, 1)$, and hence X is not indecomposable in this case. Suppose that some completion $X_i = \hat{X}_{\mathfrak{P}_i}$ contains a summand of rank $(2, 1, 1)$, so that it contains a complementary summand of rank $(1, 3, 1)$. This complementary summand must contain a summand of rank $(0, 1, 0)$, and hence a summand of rank $(1, 2, 1)$.

Suppose, then, that X_i contains no summand of rank $(2, 1, 1)$. Since $c+1>c$, X_i must contain a summand of rank $(1, 1, 0)$ or of rank $(1, 0, 0)$, and hence a complementary summand of rank $(2, 3, 2)$ or of rank $(2, 4, 2)$. If this complementary summand contains a summand of rank $(1, 1, 0)$ and $(0, 1, 1)$, then we are done, so suppose that it does not. Then in the case of the summand of rank $(2, 3, 2)$, it must contain a summand of rank $(0, 1, 0)$ and a complementary summand of rank $(2, 2, 2)$. Since this complementary summand cannot contain a summand of rank $(2, 1, 1)$, an easy argument shows that it must contain a summand of rank $(1, 1, 1)$, so that X_i contains a summand of rank $(1, 2, 1)$. In the case of the complementary summand of rank $(2, 4, 2)$, it must also contain a summand of rank $(0, 1, 0)$ and hence a complementary summand of rank $(2, 3, 2)$, which we just showed must contain a summand of rank $(1, 2, 1)$. By Corollary 3.3, X must have a summand of rank $(1, 2, 1)$. This completes the proof of the claim for $c=2$.

If $c \geq 3$, then we claim that X has a summand of rank $(2, 2, 2)$, so that X is not indecomposable in this case, either. Suppose, first, that the completion $X_i = \hat{X}_{\mathfrak{P}_i}$ contains a summand of rank $(2, 1, 1)$, and hence a complementary summand of rank $(c-1, c+1, c-1)$. If this complementary summand contains a summand of rank $(0, 1, 1)$, then X_i contains a summand of rank $(2, 2, 2)$. Otherwise, since $c+1 > c-1$, this summand must contain a summand of rank $(0, 2, 0)$ and a complementary summand of rank $(c-1, c-1, c-1)$. Since $c-1 \geq 2$, by Step 1 this summand contains a summand of rank $(2, 2, 2)$.

Suppose, then, that X_i contains no summand of rank $(2, 1, 1)$. If it contains a summand of rank $(1, 1, 0)$, then it contains a complementary summand of rank $(c, c+1, c)$, but it contains no summand of rank $(1, 0, 1)$. If the summand of rank $(c, c+1, c)$ contains a summand of rank $(0, 1, 0)$, then it contains a complementary summand of rank (c, c, c), and so it contains a summand of rank $(2, 2, 2)$, by Step 1. Otherwise, it must contain summands of rank $(1, 1, 0)$ and $(0, 1, 1)$, and hence a complementary summand of rank $(c-1, c-1, c-1)$, so that again it contains a summand of rank $(2, 2, 2)$, by Step 1.

Finally, suppose that X_i does not contain a summand of rank $(2, 1, 1)$ or $(1, 1, 0)$. Since $c+1 > c$, it must contain a summand of rank $(1, 0, 0)$ and hence a complementary summand of rank $(c, c+2, c)$. Since $c+2 > c$ and this summand cannot contain a summand of rank $(1, 1, 0)$, it must contain a summand of rank $(0, 2, 0)$, and hence a complementary summand of rank (c, c, c). Again, by Step 1, X_i must contain a summand of rank $(2, 2, 2)$. By Corollary 3.3, X must have a summand of rank $(2, 2, 2)$.

This completes the proof of the claim for $c \geq 3$, and hence Step 8 is finished.

Clearly Steps 1 through 8 handle all possible cases, which completes the proof of (v) and hence of the whole theorem. □

Since, for any odd prime p, there exists an integer n relatively prime to p such that $p\mathbb{Z}(\zeta_n)$ splits as the product of three or more prime ideals, we get the following corollary.

Corollary 3.6. *For any odd prime p, there exists an abelian group G such that $\mathbb{Z}_pG$ has finite representation type, and such that $\mathbb{Z}_pG$ has an indecomposable lattice which can be embedded in $\mathbb{Z}_pG^{(4)}$ but not in $\mathbb{Z}_pG^{(3)}$.* □

4. Two examples

Example 4.1. Given a positive integer n, in this example we construct a nonabelian group G such that, for some prime p, $\mathbb{Z}_pG$ has finite representation type, but $\mathbb{Z}_pG$ has an indecomposable lattice that cannot be embedded as a sublattice of $\mathbb{Z}_pG^{(n)}$. (Note that this example is a counterexample to an erroneous claim in the remark in [7, p. 71].)

By [2, Theorem 33.6], the integral group ring $\mathbb{Z}_pG$ has finite representation type if and only if the p-Sylow subgroup of G is cyclic of order at most p^2. Choose primes q and r such that $r > 2$ and $q > n+1$, and (using Dirichlet's Theorem), choose a prime p such that $qr | (p-1)$. Then there exists a nonabelian group H of order pq. (This follows immediately from the fact that, if P is a cyclic group of order p, then the automorphism group $\mathrm{Aut}(P)$ is itself cyclic of order $p-1$, so that P has an automorphism of order q.) Let $G = H \times C_r$, where C_r is a cyclic group of order r. Since $|G| = pqr$, it follows that $\mathbb{Z}_pG$ has finite representation type.

By [7, Theorem 3.10], $\mathbb{Z}_pG$ embeds as an order in

$$\mathbb{Q} \oplus \mathbb{Q}(\zeta_q) \oplus \mathbb{Q}(\zeta_r) \oplus \mathbb{Q}(\zeta_{qr}) \oplus \mathbb{Q}(\zeta_p) \circ C_q \oplus \mathbb{Q}(\zeta_{pr}) \circ C_q ,$$

where C_q is a cyclic group of order q, and $\mathbb{Q}(\zeta_p) \circ C_q$ and $\mathbb{Q}(\zeta_{pr}) \circ C_q$ denote *skew group algebras* (see [7, Notation 3.2]). In fact, let Λ be the pullback of the maps

$$\mathbb{Z}_p(\zeta_r) \oplus \mathbb{Z}_p(\zeta_{qr}) \xrightarrow{f} \mathbb{Z}_p(\zeta_r)/\langle p \rangle \oplus \mathbb{Z}_p(\zeta_{qr})/\langle p \rangle \xleftarrow{g} \mathbb{Z}_p(\zeta_{pr}) \circ C_q ,$$

where $\mathbb{Z}_p(\zeta_{pr}) \circ C_q$ is a skew group order in the algebra $\mathbb{Q}(\zeta_{pr}) \circ C_q$. (It follows from [7, Corollary 4.10], that $\mathbb{Z}_p(\zeta_{pr}) \circ C_q / \langle p \rangle \cong \mathbb{Z}_p(\zeta_r)/\langle p \rangle \oplus \mathbb{Z}_p(\zeta_{qr})/\langle p \rangle$.) Then by [7, Theorem 3.10], Λ is a direct summand of the group ring $\mathbb{Z}_pG$. We construct an indecomposable Λ-lattice that cannot be embedded as a sublattice of $\Lambda^{(q-2)}$, which will complete the example.

Let us set $R = \mathbb{Z}_p(\zeta_r)$; clearly Λ is an R-order. Moreover, since $r | (p-1)$, by [2, Proposition 4.34] it follows that the prime p splits completely in R. That is, $pR = \mathfrak{P}_1 \cdot \ldots \cdot \mathfrak{P}_{r-1}$, where $r - 1 \geq 2$, by assumption. Thus, the p-adic completion of Λ splits as the direct sum of the $\mathfrak{P}_j$-adic completions, $1 \leq j \leq r-1$. In addition, each of the primes $\mathfrak{P}_j$ split completely in $\mathbb{Z}_p(\zeta_{qr}) = R(\zeta_q)$. (Again, this follows because $q | (p-1)$.) Let us write $\mathfrak{P}_1 R(\zeta_q) = \mathfrak{Q}_1 \cdot \ldots \cdot \mathfrak{Q}_{q-1}$. (We shall not need to factor the other $\mathfrak{P}_j R(\zeta_q)$.)

Let X_1 be the $\hat{\Lambda}_{\mathfrak{P}_1}$-lattice constructed as follows. The completion $\hat{R}(\zeta_q)_{\mathfrak{P}_1}$ is isomorphic to the direct sum of completions $\bigoplus_{k=1}^{q-1} \hat{R}(\zeta_q)_{\mathfrak{Q}_k}$. By [7, Corollary 4.12], for each index k, $1 \leq k \leq q-1$, there exists an indecomposable $\hat{\mathbb{Z}}(\zeta_{pr})_{\mathfrak{P}_1} \circ C_q$-lattice V_k such that there is an indecomposable $\hat{\Lambda}_{\mathfrak{P}_1}$-lattice L_k consisting of the pullback of $\hat{R}(\zeta_q)_{\mathfrak{Q}_k}$ and V_k mapping onto a simple $\hat{\Lambda}_{\mathfrak{P}_1}$-module. Let $X_1 = L_1 \oplus \cdots \oplus L_{q-1} \oplus (\hat{R}_{\mathfrak{P}_1})^{(q-1)}$.

For each index j, $2 \leq j \leq r-1$, let X_j be the $\hat{\Lambda}_{\mathfrak{P}_j}$-lattice constructed as follows. By [7, Corollary 4.12], there exists an indecomposable $\hat{\mathbb{Z}}_{\mathfrak{P}_j}(\zeta_{pr}) \circ C_q$-lattice W_j such that there is an indecomposable $\hat{\Lambda}_{\mathfrak{P}_j}$-lattice M_j consisting of the pullback of $\hat{R}_{\mathfrak{P}_j}$ and W_j mapping onto a simple $\hat{\Lambda}_{\mathfrak{P}_j}$-module. Let $X_j = \hat{R}(\zeta_q)_{\mathfrak{P}_j} \oplus M_j^{(q-1)}$.

By [7, Corollary 4.4], there exists a Λ-lattice N such that $\hat{N}_{\mathfrak{P}_j} \cong X_j$ for each index j, $1 \le j \le r-1$. (Note that $\operatorname{rank}(\mathbb{Q}(\zeta_r)\cdot N) = q-1$, $\operatorname{rank}(\mathbb{Q}(\zeta_{qr})\cdot N) = 1$, and $\operatorname{rank}(\mathbb{Q}(\zeta_{pr})\circ C_q \cdot N) = q-1$.) A straightforward argument shows that N is indecomposable, and clearly N cannot be embedded as a sublattice of $\Lambda^{(q-2)}$.

Example 4.2. Given a positive integer n, in this example we construct an abelian group G such that the integral group $\mathbb{Z}G$ has finite representation type, but $\mathbb{Z}G$ has an indecomposable lattice that cannot be embedded as a sublattice of $\mathbb{Z}G^{(2^n-1)}$.

By [2, Theorem 33.6], the integral group ring $\mathbb{Z}G$ has finite representation type if and only if every Sylow subgroup of G is cyclic of at most prime-squared order. For this example, let G be a cyclic group of order $p_1^2 \cdots p_n^2$, where $p_1, \ldots, p_n$ are distinct, odd primes. From Proposition 2.1, it follows that $\mathbb{Z}G$ embeds as an order in $\bigoplus_m \mathbb{Q}(\zeta_m)$, where m ranges over all divisors of $p_1^2 \cdots p_n^2$.

For each integer j, $1 \le j \le n$, let $r_j = p_1^2 \cdots p_j^2$, and set $r_0 = 1$. Fixing an integer i, $1 \le i \le n$, let Λ_i denote the special quasi-triad constructed from the localizations $\mathbb{Z}_{p_i}(\zeta_{r_{i-1}})$, $\mathbb{Z}_{p_i}(\zeta_{r_{i-1}p_i})$ and $\mathbb{Z}_{p_i}(\zeta_{r_{i-1}p_i^2})$, as in Corollary 2.6. Note that Λ_i is a summand of the localization $\mathbb{Z}_{p_i}G$. Clearly we can find a $\mathbb{Z}_{p_i}G$-lattice X_i such that, for each integer j, $1 \le j < n$, $\operatorname{rank}(\mathbb{Q}(\zeta_{r_{j-1}})\cdot X_i) = 2^{n-j+1}$ and $\operatorname{rank}(\mathbb{Q}(\zeta_{r_{j-1}p_j})\cdot X_i) = 2^{n-j}$, while for all other divisors m of $|G|$, $\operatorname{rank}(\mathbb{Q}(\zeta_m)\cdot X_i) = 0$. (Thus, $\operatorname{rank}(\mathbb{Q}(\zeta_{r_{j-1}p_j^2})\cdot X_i) = \operatorname{rank}(\mathbb{Q}(\zeta_{r_j})\cdot X_i) = 2^{n-j}$.) Moreover, if $p_i\mathbb{Z}(\zeta_{r_{i-1}}) = \mathfrak{P}_1 \cdot \ldots \cdot \mathfrak{P}_g$, then the p_i-adic completion of Λ_i splits as the direct sum of the $\mathfrak{P}_k$-adic completions, $1 \le k \le g$. Using Proposition 3.1, we assume, in addition, that each $\mathfrak{P}_k$-adic completion $(\hat{\Lambda}_i)_{\mathfrak{P}_k} \cdot X_i$ is the direct sum of 2^{n-i} indecomposable $(\hat{\Lambda}_i)_{\mathfrak{P}_k}$-lattices, each of rank $(2, 1, 1)$. (This agrees with the above assumption that $\operatorname{rank}(\mathbb{Q}(\zeta_{r_{i-1}})\cdot X_i) = 2^{n-i+1} = 2\cdot 2^{n-i}$ and $\operatorname{rank}(\mathbb{Q}(\zeta_{r_{i-1}p_i})\cdot X_i) = \operatorname{rank}(\mathbb{Q}(\zeta_{r_{i-1}p_i^2})\cdot X_i) = 2^{n-i}$.)

Since the ranks of the lattices X_i, as i ranges from 1 to n, are all the same, by [12, Theorem 4.22] there exists a $\mathbb{Z}G$-lattice M such that the localization $M_{p_i} \cong X_i$ for each index i. By an easy induction argument and the construction of M given above, it follows that M is an indecomposable lattice. Moreover, rank $(\mathbb{Q}(\zeta_{r_0})\cdot M) = \operatorname{rank}(\mathbb{Q}(\zeta_{r_0})\cdot X_1) = 2^n$, so that M does not embed as a sublattice of $\mathbb{Z}G^{(2^n-1)}$. (We write $\mathbb{Q}(\zeta_{r_0})$, instead of just $\mathbb{Q}$, to emphasize the fact that $\mathbb{Q}(\zeta_{r_0})$ is a summand of the ring $\mathbb{Q}G \cong \bigoplus_m \mathbb{Q}(\zeta_m)$.)

Acknowledgment

I thank the referee for giving this paper a careful reading, catching several typographical errors, and suggesting improvements to the exposition.

References

[1] L. Chouinard and S. Wiegand, Ranks of indecomposable modules over one-dimensional rings, II, J. Pure Appl. Algebra 74 (2) (1991) 119–147.
[2] C. Curtis and I. Reiner, Methods of Representation Theory with Applications to Finite Groups and Orders, Vol. I (Wiley-Interscience, New York, 1981).
[3] J. Haefner and L. Klingler, Special quasi-triads and integral group rings of finite representation type, I, J. Algebra, to appear.
[4] J. Haefner and L. Klingler, Special quasi-triads and integral group rings of finite representation type, II, J. Algebra, to appear.
[5] J. Haefner and L. Levy, Commutative orders whose lattices are direct sums of ideals, J. Pure Appl. Algebra 50 (1988) 1–20.
[6] A. Jones, Groups with a finite number of indecomposable integral representations, Michigan Math. J. 10 (1963) 257–261.
[7] L. Klingler, Integral representations of groups of square-free order, J. Algebra 129 (1) (1990) 26–75.
[8] L. Levy, Mixed modules over $\mathbb{Z}G$, G cyclic of prime order, and over related Dedekind pullbacks, J. Algebra 71 (1981) 62–114.
[9] L. Levy, Modules over Dedekind-like rings, J. Algebra 93 (1985) 1–116.
[10] L. Levy, $\mathbb{Z}G_n$-modules, G_n cyclic of square-free order n, J. Algebra 93 (1985) 354–375.
[11] S. Perlis and G. Walker, Abelian group algebras of finite order, Trans. Amer. Math. Soc. 68 (1950) 420–426.
[12] I. Reiner, Maximal Orders (Academic Press, New York, 1975).